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Padé-Type Approximants of Markov and Meromorphic Functions

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We study the question of convergence of Padé and Padé-type approximants to functions meromorphic in a domain. As an example we investigate in detail the case of functions of the form

$$\hat{\mu}(z) + \sum_{j=1}^{\infty} \frac{a_j}{z - b_j}$$

where $\hat{\mu}(z)$ is a Markov function.

1. INTRODUCTION

1.1. *Background*

A classic theorem by A. A. Markov states that the diagonal Padé approximants constructed at infinity to a Markov function

$$\hat{\mu}(z) = \int_{\Delta} \frac{d\mu(t)}{z - t}$$

converge to this function locally uniformly in $\overline{\mathbf{C}} \backslash \Delta$, if μ is a positive measure with compact support on the real axis $\mathbf{R}$, Δ is the minimal

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segment of $\mathbf{R}$ containing the support of μ, and $\bar{\mathbf{C}}$ is the extended complex plane.

Rakhmanov ([10], Theorem 2) showed that the natural analogue of Markov's theorem does not necessarily hold if we perturb $\hat{\mu}$ by a rational function. For example, if we consider the function $f(z) = \hat{\mu}(z) + 1/(z-b)$ we do not in general have local uniform convergence outside $\Delta \cup \{b\}$; poles of the corresponding Padé approximants may cluster in the domain of analyticity of $f(z)$ outside of Δ. Hence, for a function of the form $f(z) = \hat{\mu}(z) + r(z)$, where $r(z)$ is a rational function, we have to make a restriction either on μ or on $r(z)$ to get results on local uniform convergence outside the union of Δ and the poles of $r(z)$. In Section 4 we discuss some results by Gončar and Rakhmanov in this direction.

It follows that if we want to get convergence results for the whole class of functions $f(z) = \hat{\mu}(z) + r(z)$ we have to consider rational approximants different from the Padé approximants. The most natural way is to fix beforehand some or all poles of the approximants. Gončar in [4], Theorem 2 considers approximants where all poles except k are preassigned, where k is the degree of the denominator of $r(z)$. The preassigned poles in [4] are the zeros of the Chebyshev polynomials or any other system of classical orthogonal polynomials.

All the results mentioned above are about the case when $f(z)$ has finitely many poles in the domain $\bar{\mathbf{C}} \backslash \Delta$. In this paper we prove similar convergence results for functions with infinitely many poles in $\bar{\mathbf{C}} \backslash \Delta$. To do this we have to consider rational approximants where a growing number of poles are preassigned (Section 2) or all poles are preassigned (Section 3).

1.2. *Padé Type Approximants*

We now introduce the rational approximants with preassigned poles referred to in Section 1.1.

Let $f(z)$ be a function which is holomorphic in a punctured neighbourhood of infinity and has a removable singularity or a pole at infinity. Let n be a nonnegative integer and let $u_k(z)$, $\not\equiv 0$, be a given polynomial of degree k, where $0 \leqslant k \leqslant n$. We introduce

$$Q_n(z) = u_k(z)\, \tilde{Q}_{n-k}(z),$$

where $\tilde{Q}_{n-k}(z) \not\equiv 0$ is a polynomial of degree at most $n-k$. We determine $\tilde{Q}_{n-k}(z)$ and a polynomial $P_n(z)$ so that

$$Q_n(z)\, f(z) - P_n(z) = \mathcal{O}(z^{-n+k-1}), \qquad \text{as} \quad z \to \infty, \tag{1}$$

where the right-hand side denotes a power series in z^{-1} with lowest order term of degree $n-k+1$ or higher. This means, if $k < n$, that the $n-k+1$ coefficients of $\tilde{Q}_{n-k}(z)$ are determined by solving a system of $n-k$ linear equations. These equations are obtained by expanding $Q_n(z)\, f(z)$ in a Laurent series around infinity and requiring that the coefficients of z^{-j}, $1 \leqslant j \leqslant n-k$, are zero. After that $P_n(z)$ is determined as the polynomial part of this expansion of $Q_n(z)\, f(z)$.

Definition. The function $P_n(z)/Q_n(z)$ above is the nth *diagonal Padé type approximant* (PTA) at infinity of $f(z)$ *with preassigned poles at the zeros of* $u_k(z)$. For convenience we shall also refer to this approximant as the PTA of *order n* with preassigned poles at the zeros of $u_k(z)$. If the zeros of $u_k(z)$ are $a_1, a_2, ..., a_k$ we also refer to it as the PTA with preassigned poles at the points $a_1, a_2, ..., a_k$. If $k = 0$, $P_n(z)/Q_n(z)$ is the nth *diagonal Padé approximant* (PA) at infinity of $f(z)$.

The definition of PTAs means that k poles are preassigned and $n-k$ poles are free and determined by the interpolation condition (1). If $f(z)$ has a removable singularity at infinity $P_n(z)$ has degree at most n. This is the case usually studied and it justifies the name diagonal PAs.

Note that $P_n(z)/\tilde{Q}_{n-k}(z)$ is the diagonal PA of order $n-k$ of $f(z)\, u_k(z)$. It follows that $P_n(z)/Q_n(z)$ is uniquely determined by (1) and that $P_n(z)$ is uniquely determined by $f(z)$ and $Q_n(z)$.

We have defined PTAs with interpolation at one point, infinity. It is also possible to define PTAs-with interpolation at another point, for instance zero, or with interpolation at different points. The latter possibility leads, for $k = 0$, to multipoint PAs. Finally, it is possible to define PTAs corresponding to the non-diagonal case (see [7]).

We want to make some remarks on the early history of PTAs and on the name PTA. For $k = n$ (all poles preassigned) they were studied by Walsh in 1935 [13] with interpolation at different points. He did not use the name PTA but referred to the method as rational interpolation. Since all poles were preassigned he considered it as a natural generalization of polynomial interpolation where all poles are preassigned to be at infinity. In 1975 Gončar [4], [5], inspired by de Montessus de Ballore's theorem, considered rational interpolation with some free and some fixed poles, using the name generalized PAs or ω-PAs. In the 70's PTAs became a common tool in numerical analysis, inspired, in particular, by work by Brezinski (see the references [29], [65] and [126] in [2]). Brezinski also introduced the name PTA. It would be natural to use the name rational interpolants instead of PTAs, especially in the case $k = n$. However, in the opinion of the authors, the close connection to PAs for $k < n$ justifies the name PTAs.

It is well-known that a great diasadvantage with PAs is that it is often impossible to control the location of the poles. In this respect, of course, PTAs offer an attractive alternative. This paper is one in a series of our papers devoted to showing the usefulness of PTAs both in numerical analysis and as a theoretical tool. A first paper in this series is [1], and some further papers are under preparation.

The plan of this paper is as follows. Section 2 and 3 contain the main new results. In Section 2 we prove some results on convergence with geometric rate for PTAs (Theorem 1 and 1′) and PAs (Theorem 2) of Markov functions. In Section 3 we prove a theorem (Theorem 3) with a weaker form of geometric rate of convergence for a more general class of meromorphic functions. Section 4, finally, contains a more detailed discussion of the results by Gončar and Rakhmanov mentioned in Section 1.1, and of the connection of their results to the results of this paper.

2. PTAS OF MARKOV FUNCTIONS

2.1. *Introduction*

Let f be a Markov function (Markov–Stieltjes function)

$$f(z)=\int_S \frac{d\mu(t)}{z-t} \tag{2}$$

where S is the union of a compact interval Δ of the real line $\mathbf{R}$ and a finite or countable set $a_1, a_2, \ldots$ of points in $\mathbf{R}\setminus\Delta$ which may cluster only at the endpoints of Δ,

$$S=\Delta\cup\{a_1, a_2, \ldots\}. \tag{3}$$

We assume that μ is a finite positive measure whose support is an infinite subset of S. Towards the end of this introduction we discuss the importance of measures μ supported by sets S of the form (3).

Let $\pi_n(z)$ denote the diagonal PA of order n at infinity of the function f. If μ does not have any point mass at any of the points $a_1, a_2, \ldots$, i.e. if the support of μ is a subset of Δ, then Markov's theorem says that π_n converges to f, as n tends to infinity, locally uniformly (on compact subsets) in $\overline{\mathbf{C}}\setminus\Delta$. In fact, the convergence holds with geometric rate, locally uniformly in $\overline{\mathbf{C}}\setminus\Delta$, where the degree of convergence is measured by means of $g_\Delta(z)$, Green's function of $\overline{\mathbf{C}}\setminus\Delta$ with pole at infinity.

We want to prove a similar convergence result in $\overline{\mathbf{C}}\setminus S$ even if the measure μ has point masses at $a_1, a_2, \ldots$. We prove such convergence results both for Padé and Padé type approximants (Theorem 2 and 1, respectively) and it turns out that PTAs provide convergence for a larger class of measures μ than PAs. In the case of PTAs the methods and results are related to [6].

We now turn to the explanation of the relevance of measures μ supported by sets S of the form (3). A Jacobi matrix is a tridiagonal matrix

$$\begin{pmatrix} v_0 & h_0 & & \\ h_0 & v_1 & h_1 & \\ & h_1 & v_2 & h_2 \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

where all entries outside the three diagonals are zero, v_n are real and h_n positive numbers. Associated to the Jacobi matrix is the sequence $\{Q_n\}$ of polynomials defined by $Q_{-1}=0$, $Q_0=1$ and, for $n=0, 1, ...$,

$$zQ_n(z)=h_n Q_{n+1}(z)+v_n Q_n(z)+h_{n-1} Q_{n-1}(z).$$

If Q_n is the nth orthonormal polynomial of a positive measure μ supported by $[-1, 1]$, then, for a large class of measures μ (see [8], p. 32)

$$h_n \to \tfrac{1}{2} \qquad \text{and} \qquad v_n \to 0, \qquad \text{as} \quad n \to \infty. \tag{4}$$

Conversely, if there is given a Jacobi matrix satisfying (4), then the associated polynomials $\{Q_n\}$ are the orthogonal polynomials of a positive measure μ supported by a set S of the form (3) where $\Delta=[-1, 1]$ and $a_1, a_2, ...$ are finitely or countably many real numbers outside $[-1, 1]$ which can cluster only at the endpoints of this interval (see [9], p. 87); the class of such measures is called Nevai's class.

2.2. *Statement of Results*

Given the sequence $a_1, a_2, ...$ we introduce

$$u_k(z)=(z-a_1)(z-a_2)\cdots(z-a_k), \qquad \text{for} \quad k=1, 2, ... \tag{5}$$

For $k=0$ we put $u_k(z)=1$.

Let Δ be a compact interval on $\mathbf{R}$ and let $a_1, a_2, ...$ be a finite or countable sequence of points in $\mathbf{R}\backslash\Delta$, distinct or not, which may cluster only at the endpoints of Δ. We make the following assumption:

$$\left\{\begin{array}{l}\text{The subsequence of } a_1, a_2, ... \text{ consisting of points located} \\ \text{to the right of } \Delta \text{ is a decreasing sequence, and the subsequence} \\ \text{of points to the left of } \Delta \text{ is an increasing sequence.}\end{array}\right. \tag{6}$$

The condition (6) means that the points $a_1, a_2, ...$ are chosen from the sides. It guarantees that $u_k(z)$ defined by (5) is real and does not change sign on S, and this is the property we need.

Let $k = k(n)$ depend on n and assume that $k(n) \to \infty$, as $n \to \infty$, if the sequence $a_1, a_2, \ldots$ is infinite. If the sequence is finite we assume that $k(n)$ equals the number of points in the sequence if n is large.

In Section 2.3 we shall prove the following theorems.

Theorem 1. *Let $a_1, a_2, \ldots$ and $k = k(n)$ satisfy the conditions above. Assume that*

$$\frac{k(n)}{n} \to 0 \qquad \text{as} \quad n \to \infty.$$

Let $f(z)$ be given by (2) and let $P_n(z)/Q_n(z)$ be the PTA of order n of $f(z)$ with preassigned poles at the points $a_1, \ldots, a_k(n)$. Then

$$\limsup_{n \to \infty} \left| f(z) - \frac{P_n(z)}{Q_n(z)} \right|^{1/(2n)} \leqslant e^{-g_\Delta(z)},$$

locally uniformly in $\overline{\mathbf{C}} \setminus S$, where $g_\Delta(z)$ is Green's function of $\overline{\mathbf{C}} \setminus \Delta$ with pole at infinity and S is given by (3).

We can also get a convergence result (which is specific for PTAs) in the domain $\overline{\mathbf{C}} \setminus \Delta$ if we consider instead of $f(z)$ the function $f(z)\, u_{k(n)}(z)$, where $u_{k(n)}(z)$ is given by (5).

Theorem 1′. *Under the conditions of Theorem 1 we have*

$$\limsup_{n \to \infty} \left| f(z)\, u_k(z) - \frac{P_n(z)}{\tilde{Q}_{n-k}(z)} \right|^{1/(2n)} \leqslant e^{-g_\Delta(z)},$$

locally uniformly in $\overline{\mathbf{C}} \setminus \Delta$, where $k = k(n)$ and $\tilde{Q}_{n-k}(z) = Q_n(z)/u_k(z)$.

Remark 1. As was pointed out above the condition (6) quarantees that $u_k(z)$ is real and does not change sign on S, and this is the crucial property which we need in the proofs. So, actually, we can take this property as the condition on the points $a_1, a_2, \ldots$ to get more general results. For instance, we can replace (6) by the condition that each point $a_1, a_2, \ldots$ has even multiplicity. In that case the zeros of $u_k(z)$ at $a_1, a_2, \ldots$ shall have the corresponding multiplicities. It is also easy to check in the proof that we can replace $f(z)$ by $f(z) + r(z)$ where $r(z)$ is a rational function whose denominator is real and does not change sign on S. In this case $P_n(z)/Q_n(z)$ shall of course be the PTA of $f(z) + r(z)$, and the denominator of $r(z)$ shall be a factor of $u_k(z)$ if n is large. One consequence of this is that the measure μ may be complex at a finite number of the points $a_1, a_2, \ldots$. Observe that $r(z)$ may have poles also on Δ and outside $\mathbf{R}$. A consequence of this is that if a subinterval of Δ does not intersect the support of μ, an even number of the points $a_1, a_2, \ldots$ may belong to this subinterval.

We now formulate the following result for PAs. It corresponds to the case $k=k(n)=0$.

THEOREM 2. *Let Δ be a compact interval on $\mathbf{R}$ and let $a_1, a_2, \ldots$ be a finite or countable sequence of points in $\mathbf{R}\backslash\Delta$ which may cluster only at the endpoints of Δ. Let $P_n(z)/Q_n(z)$ be the diagonal PA of order n of the function $f(z)$ given by* (2). *Then the limit relation in Theorem* 1 *holds for $P_n(z)/Q_n(z)$ locally uniformly in $\overline{\mathbf{C}}\backslash S$.*

Remark 2. The results about PTAs in Theorem 1 and 1′ are of interest despite of Theorem 2. First, Theorem 1 and 1′ hold also in the more general situation described in Remark 1. Second, if $a_1, a_2, \ldots$ are masspoints of μ it is natural to preassign poles exactly at these points.

Remark 3. Theorem 2 may be deduced from the theorem by Rakhmanov mentioned in Section 4.1. Our proof, however, is simpler and more direct.

2.3. *Proofs*

We start with the proof of Theorem 1′.

Proof of Theorem 1′. Remember that $k=k(n)$.

$$
\begin{aligned}
Q_n(z)\, f(z) = Q_n(z) \int_S \frac{d\mu(t)}{z-t} &= \int_S \frac{[Q_n(z)-Q_n(t)]+Q_n(t)}{z-t}\, d\mu(t) \\
&= \int_S \frac{Q_n(z)-Q_n(t)}{z-t}\, d\mu(t) + \int_S \frac{Q_n(t)}{z-t}\, d\mu(t) \\
&= P_n(z) + \int_S \frac{Q_n(t)}{z-t}\, d\mu(t),
\end{aligned}
$$

where $P_n(z)$ is a polynomial of degree at most $n-1$. In fact, $P_n(z)$ is the polynomial part of $Q_n(z)\, f(z)$ and (1) holds. So we have

$$
Q_n(z)\, f(z) - P_n(z) = \int_S \frac{\tilde{Q}_{n-k}(t)\, u_k(t)}{z-t}\, d\mu(t). \tag{7}
$$

Note that $u_k(t)$ is real and does not change sign on S. This means that $u_k(t)\, d\mu(t)$ is a positive measure on S (we can change the sign of $u_k(t)$ if necessary). From (7) and (1) we conclude that $\tilde{Q}_{n-k}(z)$ is an orthogonal polynomial with respect to the measure $u_k(t)\, d\mu(t)$ and that $\tilde{Q}_{n-k}(z)$ has degree $n-k$. By multiplying numerator and denominator in the PTA by a suitable constant, we may normalize so that $\tilde{Q}_{n-k}(z)$ is orthonormal.

Furthermore, it we divide by $\tilde{Q}_{n-k}(z)$ in (7) we get, because of the orthogonality of $\tilde{Q}_{n-k}(z)$,

$$u_k(z)\, f(z) - \frac{P_n(z)}{\tilde{Q}_{n-k}(z)} = \frac{1}{\tilde{Q}_{n-k}(z)} \int_S \frac{\tilde{Q}_{n-k}(t)\, u_k(t)}{z-t}\, d\mu(t)$$

$$= \frac{1}{\tilde{Q}^2_{n-k}(z)} \int_S \frac{[\tilde{Q}_{n-k}(z) - \tilde{Q}_{n-k}(t)] + \tilde{Q}_{n-k}(t)}{z-t}$$

$$\times \tilde{Q}_{n-k}(t)\, u_k(t)\, d\mu(t)$$

$$= \frac{1}{\tilde{Q}^2_{n-k}(z)} \int_S \frac{\tilde{Q}^2_{n-k}(t)\, u_k(t)}{z-t}\, d\mu(t). \tag{8}$$

Let us fix $\varepsilon > 0$ and consider the interval $\Delta_\varepsilon = [a-\varepsilon, b+\varepsilon]$, where $[a, b] := \Delta$. We also introduce the notation

$$A_{k(n)} = \{a_1, a_2, ..., a_{k(n)}\}.$$

Our assumptions mean that

$$\operatorname{diam}(S \backslash A_{k(n)}) \to \operatorname{diam} \Delta, \qquad \text{as} \quad n \to \infty,$$

and since $u_k(t)\, d\mu(t) \equiv 0$ on A_k, we can consider the integral in (8) as taken over Δ_ε instead of S, if n is large enough. Hence, for large n we have

$$u_k(z)\, f(z) - \frac{P_n(z)}{\tilde{Q}_{n-k}(z)} = \frac{1}{\tilde{Q}^2_{n-k}(z)} \int_{\Delta_\varepsilon} \frac{\tilde{Q}^2_{n-k}(t)\, u_k(t)}{z-t}\, d\mu(t). \tag{9}$$

Let us fix a compact set F in $\bar{\mathbf{C}} \backslash \Delta_\varepsilon$. Due to the orthonormality of $\tilde{Q}_{n-k}(z)$ we obtain

$$\max_{z \in F} \left| \int_{\Delta_\varepsilon} \frac{\tilde{Q}^2_{n-k}(t)\, u_k(t)}{z-t}\, d\mu(t) \right| \leqslant \frac{1}{\operatorname{dist}(F, \Delta_\varepsilon)}, \tag{10}$$

where dist stands for distance. Furthermore, to estimate $\tilde{Q}^2_{n-k}(z)$ we use the following inequality (see [3], Chapter III, Theorem 7.1, formula (7.8))

$$\left| \frac{1}{\tilde{Q}^2_{n-k}(z)} \right| \leqslant \frac{c}{|T^2_{n-k-1}(z)|} \left| \int_{\Delta_\varepsilon} u_k(t)\, d\mu(t) \right|, \qquad \text{for} \quad z \in F, \tag{11}$$

where $T_{n-k-1}(z)$ is the orthonormal Chebyshev polynomial of degree $n-k-1$ on Δ_ε, and c depends only on F and Δ_ε. Clearly,

$$\left| \int_{\Delta_\varepsilon} u_k(t)\, d\mu(t) \right|^{1/n} \leqslant [\operatorname{diam}(S)]^{k/n}\, [\mu(\mathbf{C})]^{1/n}.$$

By using the assumption that $k/n \to 0$, as $n \to \infty$, we get

$$[\operatorname{diam}(S)]^{k/n} \to 1$$

and

$$\limsup_{n \to \infty} \left| \int_{\Delta_\varepsilon} u_k(t)\, d\mu(t) \right|^{1/n} \leqslant 1. \tag{12}$$

For the Chebyshev polynomials on Δ_ε the following limit relation is well-known

$$|T_{n-k-1}(z)|^{1/(n-k-1)} \to e^{g_{\Delta_\varepsilon}(z)}, \qquad \text{as} \quad n-k-1 \to \infty, \tag{13}$$

uniformly on F, where $g_{\Delta_\varepsilon}(z)$ is Green's function of $\overline{\mathbf{C}} \backslash \Delta_\varepsilon$. Due to the condition $k/n \to 0$, the same limit relation is true if in the left-hand side of (13) we replace the exponent $1/(n-k-1)$ by $1/n$. Finally, from (11), (12) and (13) we get

$$\limsup_{n \to \infty} \left| \frac{1}{\tilde{Q}_{n-k}^2(z)} \right|^{1/(2n)} \leqslant e^{-g_{\Delta_\varepsilon}(z)},$$

uniformly on F. By combining the last relation with (9) and (10) we obtain

$$\limsup_{n \to \infty} \left| u_k(z)\, f(z) - \frac{P_n(z)}{\tilde{Q}_{n-k}(z)} \right|^{1/(2n)} \leqslant e^{-g_{\Delta_\varepsilon}(z)}, \tag{14}$$

uniformly on F.

Since $\varepsilon > 0$ is arbitrary and $g_{\Delta_\varepsilon}(z) \to g_\Delta(z)$, as $\varepsilon \to 0$, locally uniformly in $\overline{\mathbf{C}} \backslash \Delta$, we get from (14) the limit relation in Theorem 1′. This concludes the proof of Theorem 1′.

Proof of Theorem 1. Since $k/n \to 0$, as $n \to \infty$, we get the following estimate

$$\lim_{n \to \infty} \left| \frac{1}{u_k(z)} \right|^{1/n} \leqslant 1,$$

locally uniformly in $\overline{\mathbf{C}} \backslash S$. Because of this the limit relation in Theorem 1 follows from the limit relation in Theorem 1′. This proves Theorem 1.

Proof of Remark 1. If f is replaced by $g = f + r$, we get $Q_n g = Q_n f + Q_n r$. We treat $Q_n f$ as before, and $Q_n r$ is a polynomial if n is large and it then becomes part of the polynomial P_n.

Proof of Theorem 2. From (1) and (7) we conclude that in the case $k(n) = 0$, $Q_n(z)$ is an orthogonal polynomial with respect to the measure

$d\mu(t)$. we shall first show that the zeros of $Q_n(z)$ approach the set S as $n \to \infty$. More exactly we shall prove the following lemma.

LEMMA 1. *Let K be a compact set such that $K \subset \bar{\mathbf{C}} \backslash S$. Then K contains no zeros of $Q_n(z)$ if n is large.*

Proof of Lemma 1. Without loss of generality we may assume that all the points $a_1, a_2, \ldots$ are located to the left of Δ and that $a_1 < a_2 < \cdots$. We may also assume that each point $a_1, a_2, \ldots$ belongs to the support of μ. Note that K may intersect just a finite number of intervals $[a_j, a_{j+1}]$, $j = 1, 2, \ldots$. To prove the lemma it is sufficient to show that for each fixed index j and sufficiently large n the open interval (a_j, a_{j+1}) contains exactly one zero of $Q_n(z)$ and this zero is arbitrarily close to the endpoint a_j. The last assertion follows from the following general results (see [12], Theorem 3.3.1, 6.1.1, and 3.41.2).

Let μ be a finite, positive measure with compact support, $\operatorname{supp}(\mu) \subset \mathbf{R}$. Then we have

(A) If $\operatorname{supp}(\mu)$ is a subset of a compact interval $[a, b]$, the corresponding orthogonal polynomials $Q_n(z)$ have all the zeros in the open interval (a, b).

(B) If $x \in \operatorname{supp}(\mu)$ then an arbitrary fixed neighbourhood of x contains at least one zero of the corresponding orthogonal polynomial $Q_n(z)$ if n is large.

(C) There is at least one point of the support of μ strictly between two successive zeros of $Q_n(z)$.

This proves Lemma 1. Note that the assertion of Lemma 1 is true for some open neighbourhood of K as well.

Now we use the following result (see [11], Corollary 1.1.5; we do not formulate it in its full generality): Let μ be a finite, positive measure with compact support, $\operatorname{supp}(\mu) \subset \mathbf{R}$. If the corresponding orthonormal polynomials $Q_n(z)$ have no zeros in some open neighbourhood of a compact set K, $K \cap \operatorname{supp}(\mu) = \varnothing$, then, uniformly on K,

$$\liminf_{n \to \infty} |Q_n(z)|^{1/n} \geqslant e^{g(z)},$$

where $g(z)$ is the Green function of the domain $\bar{\mathbf{C}} \backslash \operatorname{supp}(\mu)$ with pole at infinity. We now use that $\operatorname{supp}(\mu) \subset S$ and that $S \backslash \Delta$ consists of isolated points only. This means that $g(z)$ is larger than or equal to the Green function of the domain $\bar{\mathbf{C}} \backslash S$ and that the latter function coincides with the Green function $g_\Delta(z)$ of $\bar{\mathbf{C}} \backslash \Delta$.

Now Theorem 2 follows from the error formula (8), where $k(n) = 0$.

3. PTAS OF MEROMORPHIC FUNCTIONS

3.1. *Introduction*

In Theorem 1 and 2 we got geometric rate of convergence with a factor 1/2 in the exponent $1/(2n)$ for the error $f - P_n/Q_n$. This factor came from the orthogonality property of $\tilde{Q}_{n-k}$. The factor 1/2 gives a geometric rate of convergence with speed $\exp(-2g_\Delta(z))$. The functions f in Theorem 1 and 2 are all meromorphic functions in $\mathbf{C}\backslash\Delta$ of a special kind. It is, however, possible to get convergence results for PTAs with geometric rate of convergence for all functions which are meromorphic in $\overline{\mathbf{C}}\backslash\Delta$. To prove such a result we have to preassign all poles of the PTAs. Because of that we lose the nice factor 1/2. On the other hand the result is true with $\overline{\mathbf{C}}\backslash\Delta$ replaced by any domain Ω containing infinity which is regular in the sense that $\overline{\mathbf{C}}\backslash\Omega$ is a compact set with positive logarithmic capacity such that Green's function for Ω, $g_\Omega(z)$, is continuous. More precisely we have the following result.

THEOREM 3. *Let $f(z)$ be a meromorphic function in a regular domain Ω containing infinity. Let $\{Q_n(z)\}_{n=1}^{\infty}$ be a sequence of polynomials of degree n such that*

$$\lim_{n\to\infty} |Q_n(z)|^{1/n} = e^{g_\Omega(z)}$$

locally uniformly in Ω. Let $u_k(z)$ be monic polynomials of degree $k = k(n)$ where

$$\frac{k(n)}{n} \to 0, \qquad \text{as} \quad n \to \infty,$$

such that $u_k(z)$ have zeros only in poles of $f(z)$ and the zeros of $u_k(z)$ exhaust (see below) all the finite poles of $f(z)$ (with regard to multiplicity). Let $P_n(z)/Q_{n-k}(z)\, u_k(z)$ be the PTA at infinity of order n of $f(z)$ with prescribed poles at the zeros of $Q_{n-k}(z)\, u_k(z)$. Then

$$\limsup_{n\to\infty} \left| f(z) - \frac{P_n(z)}{Q_{n-k}(z)\, u_k(z)} \right|^{1/n} \leqslant e^{-g_\Omega(z)},$$

locally uniformly in the domain of analyticity of $f(z)$.

By the assumption that $u_k(z)$ exhausts all the finite poles of $f(z)$ we mean that for any finite pole of $f(z)$, $u_k(z)$ has a zero at the pole, with the same multiplicity as the order of the pole, if n is large.

Remark 4. For analytic functions f in Ω Theorem 3 is due to Walsh ([13], p. 200, Theorem III b). By using spherical metric we may assert that the PTAs converge to $f(z)$ locally uniformly in Ω in this metric. There is also a version of Theorem 3 analogous to the version of Theorem 1 given by Theorem 1′.

Remark 5. The condition on $Q_n(z)$ in Theorem 3 is satisfied for instance if $Q_n(z)$ is an orthonormal polynomial of order n of a so called *regular* measure μ; see [11], Definition 3.1.2. All the classical orthonormal polynomials are of this type.

3.2. *Proof of Theorem* 3

We denote by $a_1, a_2, ...$ the finite poles of $f(z)$, by $g_1(z)$, $g_2(z)$, ... the corresponding principal parts of the Laurent expansion of $f(z)$ at $a_1, a_2, ...$. Let $g_0(z)$ be the polynomial part of the Laurent expansion of $f(z)$ at infinity. Put $S_m(z) = g_0(z) + g_1(z) + \cdots + g_m(z)$, and $R_m(z) = f(z) - S_m(z)$.

Put $K = \overline{\mathbb{C}} \backslash \Omega$ and take an arbitrary compact set F from the domain of analyticity of $f(z)$. Consider a cycle γ of finitely many closed curves in Ω in a prescribed, small neighbourhood of K such that the index of γ is -1 at the points of K and 0 at the points of F, i.e. γ is winding once in the negative direction around K and γ is not winding around any point of F. For sufficiently large m, R_m has poles inside of γ only, in the sense that the index of γ is -1 at the poles of R_m. We fix such an m. Since $R_m(\infty) = 0$, we obtain

$$R_m(z) = \frac{1}{2\pi i} \int_\gamma \frac{R_m(t)}{z-t}\, dt, \qquad z \in F.$$

Consider the function

$$Q_{n-k}(z)\, u_k(z)\, f(z) = Q_{n-k}(z)\, u_k(z)\, S_m(z) + Q_{n-k}(z)\, u_k(z)\, R_m(z).$$

For large n the first term in the right-hand side is a polynomial by the definition of $u_k(z)$. Furthermore, for $z \in F$,

$$\begin{aligned} &Q_{n-k}(z)\, u_k(z)\, R_m(z) \\ &\quad = \frac{1}{2\pi i} \int_\gamma \frac{[Q_{n-k}(z)\, u_k(z) - Q_{n-k}(t)\, u_k(t) + Q_{n-k}(t)\, u_k(t)]\, R_m(t)}{z-t}\, dt \\ &\quad = H_m(z) + \frac{1}{2\pi i} \int_\gamma \frac{Q_{n-k}(t)\, u_k(t)\, R_m(t)}{z-t}\, dt \end{aligned}$$

where $H_m(z)$ is a polynomial. We put

$$P_n(z) = Q_{n-k}(z)\, u_k(z)\, S_m(z) + H_m(z).$$

Then

$$(Q_{n-k} u_k f)(z) = P_n(z) + \frac{1}{2\pi i} \int_\gamma \frac{Q_{n-k}(t)\, u_k(t)\, R_m(t)}{z-t}\, dt$$

i.e. $P_n(z)$ is the polynomial part of the Laurent expansion at infinity of $(Q_{n-k} u_k f)(z)$. Hence, $P_n(z)/Q_{n-k}(z)\, u_k(z)$ is the PTA of $f(z)$ of order n with preassigned poles at the zeros of $Q_{n-k}(z)\, u_k(z)$ and we gets for $z \in F$,

$$f(z) - \frac{P_n(z)}{Q_{n-k}(z)\, u_k(z)} = \frac{1}{2\pi i Q_{n-k}(z)\, u_k(z)} \int_\gamma \frac{Q_{n-k}(t)\, u_k(t)\, R_m(t)}{z-t}\, dt. \quad (15)$$

Now we use the assumption on $Q_n(z)$, the fact that $k(n)/n$ tends to zero, and that $u_k(z)$ is a monic polynomial with zeros in finite poles of $f(z)$. This gives

$$\lim_{n \to \infty} |Q_{n-k}(z)\, u_k(z)|^{1/n} = e^{g_\Omega(z)} \quad (16)$$

locally uniformly in Ω. For $z \in F$ we get, with a constant $c = c(F, \gamma)$,

$$\left| \int_\gamma \frac{Q_{n-k}(t)\, u_k(t)\, R_m(t)}{z-t}\, dt \right| \leqslant c \cdot \max_{t \in \gamma} |(Q_{n-k} u_k R_m)(t)|$$

and using the same argument as above we obtain since n is fixed

$$\limsup_{n \to \infty} \max_{z \in F} \left| \int_\gamma \frac{(Q_{n-k} u_k R_m)(t)}{z-t}\, dt \right|^{1/n} \leqslant \max_{t \in \gamma} e^{g_\Omega(t)}.$$

Since Ω is regular, $g_\Omega(t)$ is continuous and equal to zero on K. By choosing γ close to K we get the right-hand side of the last inequality close to 1. By combining this with (15) and (16) we get Theorem 3.

4. DISCUSSION

From Gončar's paper [4], p. 557 we quote (with some changes of notation): “The construction of Padé approximants is essentially non-linear; this gives a lot of difficulty in the theory of convergence of Padé approximants. In particular, the problem as to what effect the passage from a holomorphic function f^* to the meromorphic function $f = f^* + r$ (r being a rational function) would have on convergence of these approximants has not been analytically resolved in the general case.

Below we shall consider this problem for the case when $f^* = \hat{\mu}$ where

$$\hat{\mu}(z) = \int_{\Delta} \frac{d\mu(t)}{z-t} \tag{17}$$

where μ is a non-negative measure (of bounded variation) whose support belongs to the interval $\Delta = [-1, 1]$."

We shall discuss the results by Gončar and Rakhmanov referred to in the introduction and their relation to PTAs and to the results of this paper.

4.1. *PAs*

In 1975 Gončar [4] proved the first result on the convergence of the diagonal PAs for functions of the form

$$f(z) = \hat{\mu}(z) + r(z) \tag{18}$$

where $\hat{\mu}(z)$ is given by (17) and $r(z)$ is rational. He established the following result (see [4], Theorem 1 and [10], p. 244), where μ' is the Radon–Nikodym derivative of μ with respect to Lebesgue measure. $\phi(z)$ is a function satisfying $\phi(\infty) = \infty$ which maps the complement of Δ conformably onto the exterior of unit circle. The relation of $\phi(z)$ to Green's function $g_{\Delta}(z)$ is $|\phi(z)| = \exp(g_{\Delta}(z))$.

Theorem G (Gončar). *Suppose that $\mu' > 0$ almost everywhere on Δ, that all the poles of $r(z)$ belong to $D = \overline{\mathbb{C}} \backslash \Delta$, and that $r(\infty) = 0$. Let D' be the domain D with the poles of $f(z)$ removed. Then the diagonal PAs π_n of $f(z)$ satisfy*

$$\limsup_{n \to \infty} |f(z) - \pi_n(z)|^{1/n} \leqslant \frac{1}{|\phi(z)|^2} \tag{19}$$

locally uniformly in D'.

Later Rakhmanov [10] obtained an analogous result for arbitrary measures but under a further restriction on the rational function $r(z)$. He established the inequality (19) for the diagonal PAs for the function (18) but in the case when the coefficients of $r(z)$ are real, the poles of $r(z)$ belong to D, and $r(\infty) = 0$. As mentioned in Section 1.1, he also showed that convergence does not hold for general positive measures μ and general $r(z)$; it is necessary to make a restriction either on μ or on $r(z)$.

We may relax the condition on $r(z)$ in Rakhmanov's theorem if we replace the PAs by the PTAs $P_n(z)/Q_n(z)$ of order n of $f(z)$ with preassigned poles at the zeros of $r(z)$. We may omit the assumption that the coefficients of the numerator of $r(z)$ are real (but keep it for the denominator), and we don't need the condition $r(\infty)=0$. This follows from Remark 1 in Section 2.2.

4.2. *PTAs with All Poles Preassigned*

It follows from Theorem 3 that if we preassign all poles in a proper way we get convergence for *arbitrary* functions $f(z)$ of type (18). We get the corresponding limit relation with $|\phi(z)|^{-1}$ on the right-hand side. We may choose the denominator of the PTAs as follows: $Q_n(z)=T_{n-k}(z)\,u_k(z)$ where $u_k(z)$ is the denominator of $r(z)$ and k is the degree of $u_k(z)$, and $T_{n-k}(z)$ is, for instance, any classical orthogonal polynomial.

Our main remark in this case is that in order to accelerate convergence to $f(z)$ given by (18) we may choose as $T_{n-k}(z)$ polynomials of the corresponding degree orthogonal with respect to μ, that is the denominator of ordinary PAs of the function (17). Then in the limit relation we will again have for the function (17), $|\phi(z)|^{-2}$ instead of $|\phi(z)|^{-1}$. This means that constructing the PTAs for the function (18) we ignore in a sense the rational term $r(z)$ of this function but preserve the same rate of convergence. We briefly mention here that the increasing of the rate of convergence is a consequence of the fact that in the function (17) with $d\mu(t)$ replaced by $u_k(t)\,d\mu(t)$, which is the function which we actually approximate, the weight $u_k(t)$ is an entire function. The authors intend to devote a separate paper to this question.

In this connection we mention the following theorem by Gončar ([4], Theorem 2) for functions $f(z)$ which are meromorphic in $D=\overline{\mathbb{C}}\setminus\Delta$ with k poles in D ($k<\infty$): If we fix $n-k$ poles of $Q_n(z)$ at the zeros of $T_{n-k}(z)$, where $T_{n-k}(z)$ are, for example, Chebyshev polynomials or any classical orthogonal polynomials, and the other k poles are free, then we get convergence with the rate $|\phi(z)|^{-1}$. So we see that if we fix the remaining k poles too and as $T_{n-k}(z)$ take polynomials orthogonal with respect to μ, then we may obtain the convergence rate $|\phi(z)|^{-2}$.

We also note that if the function (18) is given, the construction of the corresponding approximants become easier and easier as we pass from PAs to PTAs with finitely many preassigned poles and to PTAs with all poles preassigned.

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